§6 Differentiation

6.1 Derivative

Definition:

Let $I \subseteq \mathbb{R}$ be an interval, let $f: I \to \mathbb{R}$ and let $c \in I$.

We say that $L \in \mathbb{R}$ is the derivative of f at c if

given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in I$ with $|x-c| < \delta$, we have $\left| \frac{f(x)-f(c)}{x-c} - L \right| < \varepsilon$.

(In other words, $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = L$.)

In this case, we say that f is differentiable at c and we denote L by $f'(c)$.

Furthermore, if f is differentiable at every point whenever it is defined, f is said to be a differentiable function.

Theorem: (Differentiability $\Rightarrow$ Continuity)

If $f: I \to \mathbb{R}$ is differentiable at $c \in I$, then f is continuous at c.

proof:

(i) $\lim_{x\to c} x-c = 0$

(ii) $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c) \in \mathbb{R}$

(i), (ii) $\Rightarrow$
$$\begin{aligned}\lim_{x\to c} f(x)-f(c) &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot (x-c) \\ &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot \lim_{x\to c} x-c \\ &= f'(c) \cdot 0 \\ &= 0\end{aligned}$$

$\therefore \lim_{x\to c} f(x) = f(c)$ which means f is continuous at c.

Theorem: (Algebraic Properties)

Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$ and let $f, g: I \to \mathbb{R}$ be functions that are differentiable at c. Then,

1) $(f \pm g)'(c) = f'(c) \pm g'(c)$

2) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

3) If $g(c) \neq 0$ (Exercise: show that $g(x) \neq 0$ in some open neighborhood of c),
$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$$

proof of (2):

$$\lim_{x\to c}\frac{f(x)g(x)-f(c)g(c)}{x-c}$$

$$=\lim_{x\to c}\frac{f(x)g(x)-f(c)g(x)+f(c)g(x)-f(c)g(c)}{x-c}$$

$$=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\,g(x)+f(c)\,\frac{g(x)-g(c)}{x-c}$$

$$=f'(c)\underbrace{g(c)}_{\dagger}+f(c)g'(c)$$

† Note: g is differentiable at c

$\Rightarrow$ g is continuous at c

$\therefore \lim_{x\to c} g(x)=g(c)$

Remark:

1) If $g(x)=\alpha\in\mathbb{R}$ which is a constant function, show that $g'(c)=0$ for all $c\in\mathbb{R}$.

2) Using property (2) and putting $g(x)=\alpha$, we have $(\alpha f)'(c)=\alpha f'(c)$.

Theorem: (Chain Rule)

Let I, J be intervals in $\mathbb{R}$, let $g: I\to\mathbb{R}$ and $f: J\to\mathbb{R}$ be functions such that $f(J)\subseteq I$ and f is differentiable at $c\in J$, g is differentiable at $f(c)\in I$.

Then $g\circ f: J\to\mathbb{R}$ is differentiable at c and $(g\circ f)'(c)=g'(f(c))f'(c)$.

Idea of the proof:

$$\lim_{x\to c}\frac{g(f(x))-g(f(c))}{x-c}$$

$$=\lim_{x\to c}\boxed{\frac{g(f(x))-g(f(c))}{f(x)-f(c)}}\cdot\frac{f(x)-f(c)}{x-c}\quad (*)$$

$$=g'(f(c))f'(c)$$

But NOT correct as $f(x)-f(c)$ may equal to 0 at points in a neighborhood of c.

Therefore, we have to replace $(*)$ by something "good"!

proof:

$$\text{Let } h(y)=\begin{cases}\dfrac{g(y)-g(f(c))}{y-f(c)} & \text{if } y\in I \text{ with } y\neq f(c)\\[2ex] g'(f(c)) & \text{if } y=f(c)\end{cases}$$

Check: $\dfrac{g(f(x))-g(f(c))}{x-c}=h(f(x))\cdot\dfrac{f(x)-f(c)}{x-c}$ for all $x\in J\setminus\{c\}$.

Case 1: $f(x)\neq f(c)$, trivial!

Case 2: $f(x)=f(c)$, LHS = RHS = 0.

Furthermore, claim: $h(y)$ is continuous at $f(c)$

$\lim_{y\to c} h(y) = \lim_{y\to c} \frac{g(y)-g(f(c))}{y-f(c)} = g'(f(c)) = h(f(c))$

$\because$ g is differentiable at $y=f(c)$

Also, f is differentiable at $c \Rightarrow f$ is continuous at c

$\therefore$ $h(f(x))$ is continuous at c (Composition of continuous functions.)

i.e. $\lim_{x\to c} h(f(x)) = h(f(c)) = g'(f(c))$

$$\lim_{x\to c} \frac{g(f(x))-g(f(c))}{x-c}$$
$$= \lim_{x\to c} h(f(x)) \cdot \frac{f(x)-f(c)}{x-c}$$
$$= \lim_{x\to c} h(f(x)) \cdot \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$
$$= g'(f(c)) \cdot f'(c)$$

6.2 The Mean Value Theorem

MVT (Most Valuable Theorem !?)

Theorem:

Let I be an open interval and $f: I \to \mathbb{R}$ be a function such that f attains maximum or minimum at $c\in I$. If $f'(c)$ exists, then $f'(c)=0$.

proof:

Suppose that f attains maximum at c.

For $x\in I$, with $x<c$, we have $\frac{f(x)-f(c)}{x-c} \geq 0$

$\therefore$ $f'(c) = \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \geq 0$

Similarly, for $x\in I$, with $x>c$, we have $\frac{f(x)-f(c)}{x-c} \leq 0$

$\therefore$ $f'(c) = \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \leq 0$

$\therefore$ $f'(c) = 0$

Remark:

Solving $f'(x)=0$ is NOT enough to capture every extremum unless we know f is differentiable everywhere.

Theorem : (Rolle's Theorem)

Suppose $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b),

and $f(a)=f(b)=0$. $(*)$

Then, there exists $c\in(a,b)$ such that $f'(c)=0$.

proof :

Max-Min Theorem

$\Rightarrow$ there exists $x_M, x_m \in [a,b]$ such that $f(x_M)\geq f(x)\geq f(x_m)$ for all $x\in[a,b]$.

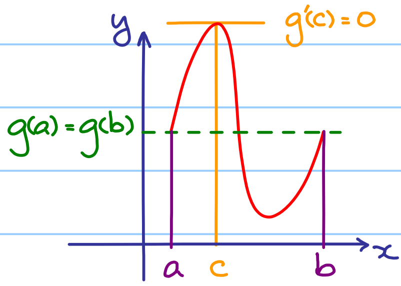
Case 1 : If either x_M or $x_m \in (a,b)$, by the previous theorem, done!

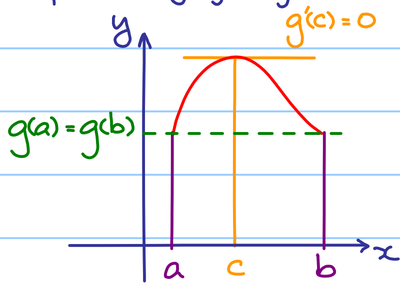
Case 2 : If both x_M and x_m lie on the boundary of $[a,b]$, then $f(x)=0$ for all $x\in[a,b]$,

then $f'(x)=0$ for all $x\in(a,b)$.

Exercise :

Prove that the result still holds true if $(*)$ is replaced by $f(a)=f(b)$.



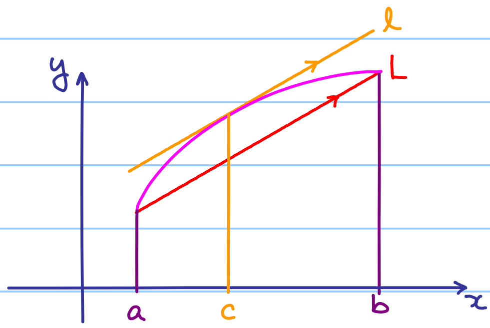


Theorem : (Mean Value Theorem)

Suppose $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b),

then there exists $c\in(a,b)$ such that $f(b)-f(a)=f'(c)(b-a)$

Geometrical Meaning :



$f'(c) = \dfrac{f(b)-f(a)}{b-a}$

slope of ℓ slope of L.

proof : Looking for $c\in(a,b)$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

i.e. looking for a solution in (a,b) of the equation

$\underline{f'(x)(b-a)-[f(b)-f(a)]}=0$.

Idea : Realize this as $g'(x)$ and apply Rolle's theorem.

Let $g(x) = f(x)(b-a) - x[f(b)-f(a)]$ (then $g'(x) = f'(x)(b-a) - [f(b)-f(a)]$)

Check : 1) g is continuous on $[a,b]$

2) g is differentiable on (a,b)

3) $g(a) = g(b) = bf(a) - af(b)$

Apply Rolle's Theorem to g, the result follows.

Theorem :

Suppose $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) and that $f'(x)=0$ for all $x\in(a,b)$.

Then f is constant on $[a,b]$.

proof : For $a<x\leq b$, note f is differentiable everywhere on (a,x)

$\Rightarrow f$ is continuous everywhere on $[a,x]$

Apply MVT, $\exists\, c\in(a,x)$ such that

$$f(x)-f(a) = \underbrace{f'(c)}_{\substack{\parallel \\ 0}}(x-a) = 0$$

assumption

i.e. $f(x)=f(a)$ for all $x\in[a,b]$.

Corollary :

Suppose that $f,g:[a,b]\to\mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b), and that $f'(x)=g'(x)$ for all $x\in(a,b)$.

Then $f=g+C$ for some constant C on $[a,b]$.

proof :

Consider $h(x) := f(x)-g(x)$.

Definition :

Let $f: I\to\mathbb{R}$ be a function defined on an interval I.

- f is said to be increasing (decreasing) if $f(x_1)\leq f(x_2)$ ($f(x_1)\geq f(x_2)$) for all $x_1,x_2\in I$ with $x_1<x_2$.
- f is said to be strictly increasing (decreasing) if $f(x_1)<f(x_2)$ ($f(x_1)>f(x_2)$) for all $x_1,x_2\in I$ with $x_1<x_2$.

Theorem:

Suppose that $f: I \to \mathbb{R}$ be differentiable on an interval I.

Then f is increasing (decreasing) on I if and only if $f'(x) \geq 0$ ($f'(x) \leq 0$) for all $x \in I$.

proof:

"$\Leftarrow$" Let $x_1, x_2 \in I$ with $x_1 < x_2$.

By assumption, f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2).

Applying MVT to f on $[x_1, x_2]$, there exists $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = \underbrace{f'(c)}_{\geq 0} \underbrace{(x_2 - x_1)}_{> 0} \geq 0$$

"$\Rightarrow$" Assume the contrary,

there exist $x_1, x_2 \in I$ with $x_1 < x_2$ but $f(x_1) > f(x_2)$.

Applying MVT to f on $[x_1, x_2]$, there exists $c \in (x_1, x_2)$ such that

$$\underbrace{f(x_2) - f(x_1)}_{< 0} = f'(c) \underbrace{(x_2 - x_1)}_{> 0}$$

$\therefore f'(c) < 0$ which contradicts to that $f'(x) \geq 0$ for all $x \in I$.

Theorem: (1st derivative check)

Let $f: [a,b] \to \mathbb{R}$ be a continuous function and let $c \in (a,b)$

If there exists $\delta > 0$ such that $(c-\delta, c+\delta) \subseteq I$ and

- $f'(x) \geq 0$ for all $x \in (c-\delta, c)$
- $f'(x) \leq 0$ for all $x \in (c, c+\delta)$

then f has a relative maximum at c.

(Remark: We do NOT assume the differentiability at c, but only the continuity at c.)

Similar result for relative minimum.

proof:

Apply the MVT twice:

case 1: if $x \in (c-\delta, c)$

case 2: if $x \in (c, c+\delta)$